DUSO Mathematics League 2015 - 2016

CONTEST #4.

SOLUTIONS

4 - 1. $\boxed{\mathbf{6}}$ Convert the heights to feet first. Then solve $18 + 1x = 12 + 2x$ to obtain $x = \mathbf{6}$.

4 - 2. $\boxed{\{\mathbf{1/2}, \mathbf{-2/3}\}}$ This equation is of the form $A^2 + B^2 = (A + B)^2$, which has solutions only if $A = 0$ or $B = 0$. Therefore, instead of expanding the brackets and proceeding to solve a quadratic equation, instead solve two linear equations to find $2x - 1 = 0 \to x = 1/2$ and $3x + 2 = 0 \to x = -2/3$. The solutions are $\{\mathbf{1/2}, \mathbf{-2/3}\}$.

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4 - 3. $\boxed{\mathbf{61}}$ Make use of the fact that a trigonometric function of an angle is the cofunction of its complement. Therefore, $A = 90 - 81 = 9$ and $B = 90 - 38 = 52$, so $A + B = \mathbf{61}$.

4 - 4. $\boxed{\mathbf{40} + \dfrac{\mathbf{32}\pi}{\mathbf{3}} + \mathbf{4}\sqrt{\mathbf{3}}}$ The rectangle has area $4 \cdot 10 = 40$. The circle has area 16π. Notice that there is overlap that must be subtracted. This circle intersects the rectangle when $x^2 + 2^2 = 16 \to x = \pm\sqrt{12}$, so the overlap is the area inside a sector of radius 4 and central angle $120°$ but outside an isosceles triangle with legs of length 4 and vertex angle $120°$. Thus, the overlap has area $\left(\frac{1}{2} \cdot \frac{2\pi}{3} \cdot 4^2 - \frac{1}{2}4^2\frac{\sqrt{3}}{2}\right) = \frac{16\pi}{3} - 4\sqrt{3}$. The area colored in the logo is $40 + 16\pi - \left(\frac{16\pi}{3} - 4\sqrt{3}\right) = \mathbf{40} + \dfrac{\mathbf{32}\pi}{\mathbf{3}} + \mathbf{4}\sqrt{\mathbf{3}}$.

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4 - 5. $\boxed{\mathbf{20} + \mathbf{4}\sqrt{\mathbf{7}}}$ Find the length BG using the Law of Cosines: $BG^2 = 12^2 + 8^2 - 2 \cdot 12 \cdot 8 \cdot \frac{1}{2} = 112$. Thus, the perimeter of $\triangle BIG$ is $20 + \sqrt{112} = \mathbf{20} + \mathbf{4}\sqrt{\mathbf{7}}$.

4 - 6. $\boxed{\dfrac{\mathbf{5}}{\mathbf{3}}}$ Use the change-of-base rule to rewrite the equations as $\dfrac{2\log 3}{\log 5}x + \dfrac{\log 2}{\log 7}y = \dfrac{3\log 3}{\log 5}$ and $\dfrac{\log 7}{\log 2}x - \dfrac{\log 5}{\log 3}y = \dfrac{2\log 7}{\log 2}$. Then multiply both sides of the first equation by $\dfrac{\log 5}{\log 3}$ and both sides of the second equation by $\dfrac{\log 2}{\log 7}$ to obtain $2x + \dfrac{\log 5}{\log 3} \cdot \dfrac{\log 2}{\log 7}y = 3$ and $x - \dfrac{\log 5}{\log 3} \cdot \dfrac{\log 2}{\log 7}y = 2$. Adding the equations yields $3x = 5$, so $x = \dfrac{\mathbf{5}}{\mathbf{3}}$.

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T-1. A lattice point is a point whose coordinates are integers. How many lattice points satisfy $x^2 + y^2 < 25$?
T-1Sol. $\boxed{69}$ Consider the first quadrant. There are four such points with $x = 1$ or $x = 2$, three lattice points with $x = 3$, and two with $x = 4$, for a total of 13. Thus, there are $4(13) = 52$ such lattice points that lie within a quadrant. There are nine on each axis, but one is the origin (and it got counted twice), so our answer is $52 + 2(9) - 1 = \mathbf{69}$.

T-2. Many positive integers have 14 positive integer factors. If they were to be listed in increasing order, the second number in the list would be N. Compute N.
T-2Sol. $\boxed{\mathbf{320}}$ A number with exactly 14 factors must be of the form p^{13} or $p^6 \cdot q$ for primes p and q. The least number of the first form is $2^{13} = 8192$. The smallest numbers of the second form are $2^6 \cdot 3 = 192$, $2^6 \cdot 5 = 320$, and $2^6 \cdot 7 = 448$. Note that $3^6 \cdot 2 = 1458$, which is far greater than N. The answer to the question is $N = \mathbf{320}$.

T-3. Compute all real values of x that solve
$(\sqrt{x+3} - \sqrt{1-x})^3 - (\sqrt{x+3} - \sqrt{1-x})^2 + 4(\sqrt{x+3} - \sqrt{1-x}) = 12$.
T-3Sol. $\boxed{\mathbf{1}}$ This problem is easier to solve if $Y = \sqrt{x+3} - \sqrt{1-x}$, because then the given equation becomes the cubic $Y^3 - Y^2 + 4Y - 12 = 0 \rightarrow (Y-2)(Y^2 + Y + 6) = 0$. The quadratic factor in Y has no real solutions, so we focus on the factor that implies $Y = 2$. Solving $\sqrt{x+3} - \sqrt{1-x} = 2 \rightarrow x + 3 = 1 - x + 4\sqrt{1-x} + 4 \rightarrow x - 1 = 2\sqrt{1-x}$, which has two solutions: $x = -3$ (which does not check) and $\mathbf{x = 1}$.

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